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A conformally covariant massless quantum field in 1 + 1 dimensions

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Abstract

In this paper, we construct a causal and conformally covariant massless free quantum field on the $(1+1)$-dimensional Minkowski spacetime. This field is of the Gupta–Bleuler type and does not suffer from infrared divergences. Although negative frequency states are used in its construction, the field has no unusual physical features: the energy–momentum tensor behaves reasonably when applied to physical states and it is also shown that particle detectors moving in the vacuum behave as expected.

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1. Introduction

In this paper we construct a causal and conformally covariant massless free quantum field on the $(1+1)$-dimensional Minkowski spacetime. This field is conformally invariant in a strong sense, without any correction term. We use for this construction a new procedure of quantization developed in the curved-space context in order to quantize fields with infrared divergence. This construction is the one which allowed to construct massless (respectively minimally coupled) fields on the de Sitter spacetime in two [DBR2] (respectively four [GRT]) dimensions and massless spin-2 field in four-dimensional de Sitter space [GGRRT]. Moreover this construction presents an additional interesting feature: the energy–momentum tensor is automatically and covariantly renormalized in our theory, since no infinite term appears in the computation of $\langle 0|T_{00}(x)|0\rangle$.

Although the classical massless free field equation is very simple and of course completely solvable, it is well known that in two dimensions the corresponding quantum field presents various complicating features such as gauge invariance and infrared divergence, and it is sometimes said that a conformally covariant massless quantum field does not exist in two dimensions (see [C] and references therein). On this account, it has been studied by several authors ([MPS, AM] and references therein).

As explained, for instance, in Wald's book [W], the key to any procedure of quantization is the definition of a Hilbert space containing the space of compact supported initial value solutions of the field equation with a distinguished subspace of positive frequency solutions. This construction can usually be reached in many ways (two-point distributions, complex structures, Hilbert basis, etc) which are logically equivalent. In our construction the Hilbert space is replaced by a Krein space and we adopt the more pedagogical point of view of Hilbert basis (the modes) which finally allows a rigorous construction of the Hilbert and Krein spaces we need. Our construction can be briefly described as follows. A more detailed and rigorous treatment will be given in sections 2 and 3. We will first, as is usual in canonical quantization, construct a set of modes ϕ_k satisfying

$$\begin{aligned} &\langle \phi_k, \phi_{k'} \rangle = \delta_{kk'} \\ &\langle \phi_k, \phi_{k'}^* \rangle = 0 \\ &\sum_k \phi_k(x)\phi_k^*(x') - \phi_k^*(x)\phi_k(x') = -\mathrm{i}\tilde{G}(x, x') \end{aligned} \tag{1}$$

where $\tilde{G} = G^{\mathrm{adv}} - G^{\mathrm{ret}}$ is the so-called commutator (although it is not a quantum but a geometrical object) and $\langle \cdot, \cdot \rangle$ the Klein–Gordon inner product. A family of modes satisfying the last relation above is said to be 'complete'. It is to be noted that the modes that we shall use in this paper (see section 2) are labelled by a discrete variable. They are related to a conformal-Killing time and not given by the usual plane waves, which are not particularly well adapted to the treatment of the conformal symmetry [DBR1]. We then construct with those modes the Hilbert space $\mathcal{H}_p$ they span, the Krein space $\mathcal{H} = \mathcal{H}_p + \mathcal{H}_p^*$ (on which a non-positive-definite but non-degenerate inner product is defined) and the Fock space $\underline{\mathcal{H}}$ over $\mathcal{H}$ [M]. The field will finally be defined via

$$\varphi(x) = \sum_k \phi_k(x) A_k + \phi_k^*(x) A_k^\dagger$$

where A_k and $A_k^\dagger$ are suitably defined operators on $\underline{\mathcal{H}}$ satisfying the canonical commutation relations (ccr):

$$[A_k, A_{k'}^\dagger] = \delta_{kk'} \qquad [A_k, A_{k'}] = 0 \qquad [A_k^\dagger, A_{k'}^\dagger] = 0.$$

It is clear from this definition and the above properties of the modes that this field satisfies the field equation and is causal. In fact, once the modes are chosen, our construction differs from the usual canonical quantization only in the choice of the representation of the canonical commutation relations: A_k and $A_k^\dagger$ are not given by the standard annihilation and creation operators on the Fock space over $\mathcal{H}_p$. Indeed, $\mathcal{H}_p$ is not Poincaré-invariant and such a field would not be covariant. The space $\mathcal{H}$, on the other hand, is Poincaré-invariant and, as proved in section 3, this invariance will be enough to guarantee the Poincaré covariance of the field we construct. Moreover, it turns out that our field is fully conformally covariant as well, in a sense to be explained in section 3.

In order to explore the physical content of the theory, we first remark that the total space $\underline{\mathcal{H}}$ contains unphysical negative norm states, as well as states which are at negative frequency with respect to both the conformal-Killing time and the usual time. We therefore need to identify the subspace $\underline{\mathcal{K}}$ of physical states. It is defined so as to contain only positive frequency states with respect to both the times and to be invariant under the action of the conformal group. The conformal invariance of $\underline{\mathcal{K}}$ plays, of course, a crucial role in the conformal covariance of the field. Thanks to the absence of negative frequency states in the physical space, we will be able to show that the presence of negative frequency terms in the definition of the field above does not yield unphysical negative energies in the sense that expressions such as $\langle k_1, \ldots, k_n|T_{00}|k_1, \ldots, k_n\rangle$ will be shown to be always positive. In addition, the vacuum of the field is indeed empty and we show that both inertial and accelerated particle detectors behave as expected in the field.

The above construction is of the Gupta–Bleuler type since the field acts on a total space which carries an indefinite inner product, and of which the physical space is a proper subspace. That such a construction should appear is not too surprising since the Lagrangian $\mathcal{L} = \sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ admits a gauge-like invariance: it is invariant when adding a constant function to ϕ. We will show that this gauge invariance is unitarily implemented on $\underline{\mathcal{H}}$.

In [MPS] a different construction of a Poincaré covariant field is proposed. A detailed comparison between the two approaches will be given in section 6. Our construction is simpler and the implementation of conformal and gauge invariance is more natural in our field.

The paper is organized as follows. In section 2, we describe the one-particle sector of the theory, identifying the space of physical states $\mathcal{K}$ and the total space $\mathcal{H}$. In section 3, the field is constructed and its invariance properties studied. In sections 4 and 5 we study some of its physical features. Section 6 is devoted to the comparison with the field of [MPS].

2. The one-particle sector

The goal of this section is to describe the modes ϕ_k and the space $\mathcal{H}_p$ they span and the Gupta–Bleuler triplet $\mathcal{N} \subset \mathcal{K} \subset \mathcal{H}$ on the level of the one-particle sector.

We use the metric $\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2$ on the $(1+1)$-dimensional Minkowski spacetime M. In light-cone coordinates $u = (u^+, u^-)$ with $u^\varepsilon = x^0 - \varepsilon x^1$, the conformally invariant Klein–Gordon equation reads

$$\frac{\partial^2}{\partial u^+ \partial u^-}\phi = 0.$$

For further reference, we recall that the embedding of the spacetime in the torus $S^1 \times S^1$, on which the conformal group $SO_0(2,2)/\pm\mathrm{Id}$ acts naturally, is realized by the mapping

$$M \ni (u^+, u^-) \mapsto \left(\exp \mathrm{i}\alpha^+ = \frac{u^+ + \mathrm{i}}{u^+ - \mathrm{i}}, \exp \mathrm{i}\alpha^- = \frac{u^- + \mathrm{i}}{u^- - \mathrm{i}}\right) \in S^1 \times S^1. \tag{2}$$

Recall that this map is into, but not onto. The infinitesimal generators of the local action of $SO_0(2,2)/\pm\mathrm{Id}$ on M are, for $\varepsilon = \pm$

$$X_{P^\varepsilon} = -\partial_{u^\varepsilon} \qquad X_{K^\varepsilon} = u^\varepsilon \partial_{u^\varepsilon} \qquad X_{L^\varepsilon} = -(u^\varepsilon)^2 \partial_{u^\varepsilon}.$$

Here X_{P^+}, X_{P^-} and $X_{K^+} - X_{K^-}$ generate the global action of the Poincaré group, whereas the X_{L^ε} generates the special conformal transformations, which act only locally on M, but globally on $S^1 \times S^1$. We note that

$$X = -\tfrac{1}{2}((X_{P^+} + X_{L^+}) + (X_{P^-} + X_{L^-})) = \tfrac{1}{2}(1 + (u^+)^2)\partial_+ + \tfrac{1}{2}(1 + (u^-)^2)\partial_-$$

is a globally time-like forward pointing conformal Killing field, the flow lines of which are the time-like hyperbolas

$$(x^1 + c)^2 - (x^0)^2 = 1 + c^2.$$

It admits a family of space-like hypersurfaces that are perpendicular to the flow lines and that foliate spacetime. They are the level surfaces of a 'conformal time' given by

$$t_c = \arctan u^+ + \arctan u^- \qquad t_c \in]-\pi, \pi[$$

and satisfying $X(t_c) = 1$. Note that $t_c = \pi - (\alpha^+ + \alpha^-)/2$, where $\alpha^\pm \in]0, 2\pi[$ are defined through (2).

We shall now construct the modes ϕ_k, not as eigenmodes with positive eigenvalue of the usual time translation generator ∂_{x_0}, but as eigenmodes with positive eigenvalue of X. Defining

$$\phi_k(u^+, u^-) = \frac{1}{2\sqrt{\pi k}} \left(\frac{u^+ + \mathrm{i}}{u^+ - \mathrm{i}}\right)^k \qquad \text{when } k > 0$$

$$\phi_k(u^+, u^-) = \frac{1}{2\sqrt{\pi |k|}} \left(\frac{u^- + \mathrm{i}}{u^- - \mathrm{i}}\right)^{|k|} \qquad \text{when } k < 0$$

one readily sees that

$$\mathrm{i}X\phi_k = |k|\phi_k.$$

Note that the sign of k distinguishes the right and left moving particles and that all these modes are at positive frequency in the usual sense of the word: their Fourier transforms are supported on the future cone. We have furthermore

$$\langle \phi_k, \phi_l \rangle = \delta_{kl} \qquad \langle \phi_k, \phi_l^* \rangle = 0 \quad \forall k, l \in \mathbb{Z}' = \mathbb{Z} \setminus \{0\}$$

where $\langle\ ,\ \rangle$ is the Klein–Gordon inner product:

$$\langle \phi, \psi \rangle = \mathrm{i} \int_{x^0=0} \phi^*(x^0, x^1) \overleftrightarrow{\partial_0} \psi(x^0, x^1)\, \mathrm{d}x^1. \tag{3}$$

In the following we also use the L^2 inner product:

$$(f, g) = \int f^*(x) g(x)\, \mathrm{d}^2 x. \tag{4}$$

The set $\phi_k,\ k \in \mathbb{Z}'$ is as such not complete as a direct computation will confirm. To complete it we need to add a so-called zero-mode [A] ϕ_0. Introducing for later reference $\psi_g = 1$, we define, in complete analogy with [DBR2]

$$\phi_0 = \psi_g + \frac{t_c}{4\mathrm{i}\pi}.$$

One can then check that the full set $\phi_k,\ k \in \mathbb{Z}$ is a complete set of modes in the sense of (1).

Let $\mathcal{H}_p$ be the Hilbert space generated by the $\phi_k,\ k \in \mathbb{Z}$: this space is *not closed* under the action of the Poincaré group and, as a consequence, the usual canonical quantization applied to this set of modes yields a non-covariant field. In order to proceed with Gupta–Bleuler quantization we therefore define the total space $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_p^*$, where $\mathcal{H}_p^*$ is the anti–Hilbert space generated by the ϕ_k^* (the Klein–Gordon inner product is *negative* definite on it). For further analysis we need a Hilbert space structure on $\mathcal{H}$, that we define through

$$\|\psi\|_{\mathcal{H}}^2 = \langle \psi_p, \psi_p \rangle - \langle \psi_n, \psi_n \rangle \qquad \psi = \psi_p + \psi_n \qquad \psi_p \in \mathcal{H}_p \quad \psi_n \in \mathcal{H}_p^*.$$

Note that $\langle \psi_n, \psi_n \rangle$ is negative and that these structures make $\mathcal{H}$ into a Krein space [B].

The space $\mathcal{H}$ so defined is rather abstract. Due to the somewhat unusual choice of the set of modes it is, in particular, not clear *a priori* that it is invariant under the Poincaré group. To prove this, we will now show that it is actually a Poincaré-invariant space of distributional solutions of the field equation. For that purpose, we first introduce the space $\mathcal{K}$, defined by

$$\mathcal{K} = \left\{ c_g \psi_g + \sum_{k \in \mathbb{Z}'} c_k \phi_k \text{ with } \sum_{k \in \mathbb{Z}'} |c_k|^2 < \infty \right\}.$$

It is readily checked that $\mathcal{H} = (\mathcal{K} + \mathcal{K}^*) \oplus \mathbb{C}t_c$. The space $\mathcal{K}$ was studied in detail in [DBR1] and we recall the results that will be needed in the following. $\mathcal{K}$ is a space of positive frequency distributional solutions of the field equation and can be written as $\mathcal{K} = \mathcal{K}^+ + \mathcal{K}^-$. Here $\mathcal{K}^\varepsilon$ is precisely the set of all the functions ψ of the variable u^ε that are boundary values of analytic functions (written ψ as well) on $\mathcal{P}^\varepsilon = \{z^\varepsilon = u^\varepsilon + \mathrm{i}y^\varepsilon;\ y^\varepsilon < 0\}$ and the derivatives of which are square integrable on $\mathcal{P}^\varepsilon$ equipped with the Lebesgue measure $\mathrm{d}\lambda$. Note that the above sum is not a direct sum because the constant functions belong both to $\mathcal{K}^+$ and $\mathcal{K}^-$. The crucial point is now that the spaces $\mathcal{K}^\varepsilon$ and hence $\mathcal{K}$ are invariant under the natural action of the conformal group on them: note that the latter acts naturally and globally on $\mathcal{P}^\varepsilon$ (but not on its boundary!) via fractional transformations. In addition, $\mathcal{K}^\varepsilon$ is equipped with a degenerate inner product defined for any $\psi^\varepsilon, \phi^\varepsilon \in \mathcal{K}^\varepsilon$ by

$$\langle \psi^\varepsilon, \phi^\varepsilon \rangle_\varepsilon = \frac{1}{4} \int_{\mathcal{P}^\varepsilon} \frac{\partial \psi_\varepsilon^*}{\partial z^\varepsilon}(z^\varepsilon) \frac{\partial \psi_\varepsilon}{\partial z^\varepsilon}(z^\varepsilon)\, \mathrm{d}\lambda(z^\varepsilon).$$

The inner product on all of $\mathcal{K}$ is then defined by

$$\langle \psi, \phi \rangle = \langle \psi_+, \phi_+ \rangle_+ + \langle \psi_-, \phi_- \rangle_-$$

with evident notations. Note that the non-uniqueness of the decomposition plays no role in this definition because the constant functions are orthogonal to all elements. This inner product is invariant under the conformal group, that is to say: the natural action of the conformal group is unitary with respect to this product. One can furthermore prove [DBR1] that, as suggested by the notation, this inner product is equal to the Klein–Gordon inner product defined by (3) when the latter makes sense. The inner product is however degenerate on $\mathcal{K}$. Let $\mathcal{N}$ be the one-dimensional subspace generated by ψ_g. It is then easy to see that $\mathcal{N}$ is the radical in $\mathcal{K}$ (the set of elements of $\mathcal{K}$ orthogonal to $\mathcal{K}$) since

$$\langle \psi_g, \psi_g \rangle = 0 \qquad \langle \psi_g, \phi_k \rangle = 0 \quad \forall k \in \mathbb{Z}' = \mathbb{Z} \backslash \{0\}.$$

It is an uncomplemented invariant subspace of $\mathcal{K}$: as a result, the representation of the conformal group on $\mathcal{K}$ is indecomposable. This is of course not an unusual situation in the presence of gauge degrees of freedom.

We are now ready to show that the space $\mathcal{H}$ is invariant under the Poincaré group. As said before, $\mathcal{K}$ and hence $\mathcal{K}^*$ are invariant under the action of the Poincaré group. Hence it is enough to prove that $\phi_0(x + a) - \phi_0(x)$ and $\phi_0(\Lambda x) - \phi_0(x)$, where Λ is the Lorentz boost, belong to $\mathcal{K} + \mathcal{K}^*$. Consider for instance the first one, it is enough to prove that the function

$$f(u^+) = \arctan(u^+ + a^+) - \arctan(u^+)$$

belongs to $\mathcal{K} + \mathcal{K}^*$. This can be done in the following way. One first remarks that f and all its derivatives vanish at infinity. As a consequence, applying the cayley transform $\mathrm{e}^{\mathrm{i}\alpha^+} = (u^+ + \mathrm{i})/(u^+ - \mathrm{i})$ one transforms f into a $\mathcal{C}^\infty$-function on the torus. Its Fourier series has rapidly decreasing coefficients. Returning to the variable u^+ this means that f can be written as

$$f = c_g \psi_g + \sum_{k>0} c_k \phi_k + \sum_{k>0} c_{-k} \phi_k^*$$

with rapidly decreasing c_k. This proves that $f \in \mathcal{K} + \mathcal{K}^*$. The case of the Lorentz boost is similar.

We have now completely described the (first quantized) Gupta–Bleuler triplet [BFH, G]:

$$\mathcal{N} \subset \mathcal{K} \subset \mathcal{H}.$$

The space $\mathcal{H}$ contains both negative frequency states and states for which $\langle \psi, \psi \rangle$ is negative: such states are to be excluded from the physical one-particle space which is therefore a proper subspace of the total space $\mathcal{H}$. We therefore choose the physical subspace to be $\mathcal{K}$, which contains only positive frequency states and on which the Klein–Gordon inner product is positive (even if it is non-definite). Each of the spaces $\mathcal{N}$, $\mathcal{K}$ and $\mathcal{H}$ is Poincaré-invariant. $\mathcal{N}$ and $\mathcal{K}$ are also conformally invariant. The space $\mathcal{H}$ is not (globally) conformally invariant but it is locally conformally invariant in the sense that the vector space spanned by the ϕ_k's is closed under the action of the conformal Lie algebra $so(2, 2)$. As we show below, this, together with the conformal invariance of $\mathcal{K}$ will suffice to guarantee the conformal invariance of the field.

3. The quantum field

We can now proceed to the canonical quantization, using the modes ϕ_k for $k \in \mathbb{Z}$. We define the quantum field through

$$\varphi(x) = \sum_k \phi_k(x) A_k + \phi_k^*(x) A_k^\dagger$$

where A_k and $A_k^\dagger$ are operators satisfying the canonical commutation relations (ccr):

$$[A_k, A_{k'}^\dagger] = \delta_{kk'} \qquad [A_k, A_{k'}] = 0 \qquad [A_k^\dagger, A_{k'}^\dagger] = 0.$$

What is new in our construction, beyond the choice of the modes and therefore of the associated spaces $\mathcal{K}$ and $\mathcal{H}$, is the representation of the ccr that we now describe. Following Minchev, one can define the Fock space $\underline{\mathcal{H}}$ upon the Krein space $\mathcal{H} = \mathcal{H}_p + \mathcal{H}_p^*$ and on it the usual creation and annihilation operators. They satisfy the commutation relations:

$$[a(\phi), a^\dagger(\psi)] = \langle \phi, \psi \rangle \qquad \forall \phi, \psi \in \mathcal{H}.$$

Letting $a_k = a(\phi_k)$ and $b_k = a(\phi_k^*)$, we can define

$$A_k = \frac{1}{\sqrt{2}}(a_k - b_k^\dagger) \qquad \text{and} \qquad A_k^\dagger = \frac{1}{\sqrt{2}}(a_k^\dagger - b_k).$$

Since $[a_k, a_k^\dagger] = 1$, $[b_k, b_k^\dagger] = -1$, this yields a representation of the ccr. As a consequence of the completeness of the modes, this field is clearly causal and satisfies the field equation. In order to prove the Poincaré and conformal covariance of the field it will be convenient to rewrite it in a basis-independent way, referring only to the space $\mathcal{H}$ in the construction, and not to the modes ϕ_k. For that purpose, we rewrite the field as follows. Using the definition of φ and A_k, one obtains:

$$\varphi(x) = \frac{1}{\sqrt{2}} \left(\sum_{k\in\mathbb{Z}} \phi_k(x) a_k - \sum_{k\in\mathbb{Z}} \phi_k^*(x) b_k + \sum_{k\in\mathbb{Z}} \phi_k^*(x) a_k^\dagger - \sum_{k\in\mathbb{Z}} \phi_k(x) b_k^\dagger \right). \tag{5}$$

Using the linearity and anti-linearity of the creators and annihilators, one finds

$$\varphi(x) = \frac{1}{\sqrt{2}}(a(p(x)) + a^\dagger(p(x)))$$

where

$$p(x) = \sum_{k\in\mathbb{Z}} \phi_k^*(x)\phi_k - \sum_{k\in\mathbb{Z}} \phi_k(x)\phi_k^*.$$

The field φ is an operator-valued distribution and the 'function' p, defined on M with values in $\mathcal{H}$, is a distribution as well. Since we are looking for a rigorous definition of the field, we begin with a rigorous definition of p in terms of the test functions. Using the above expression of p one can check that, formally at least, $\langle p(x), \psi \rangle = \psi(x)$ for any $\psi \in \mathcal{H}$. This can be smeared into

$$\langle p(f), \psi \rangle = (f, \psi) \qquad \forall \psi \in \mathcal{H} \tag{6}$$

where the two inner products involved are defined in (3) and (4) and f is a real test function. Thanks to the properties of Krein spaces, this last equation will yield an unambiguous and completely rigorous definition of p as a $\mathcal{H}$-valued distribution provided the map $\psi \mapsto (f, \psi)$ is continuous on $\mathcal{H}$. In order to prove this result, we consider the space

$$L = L^2\left(M, \frac{\mathrm{d}u^+\,\mathrm{d}u^-}{(1+(u^+)^2)(1+(u^-)^2)}\right)$$

and prove the following lemma:

Lemma . *Any ψ in $\mathcal{H}$ belongs to L and there exists a constant C such that*

$$\|\psi\|_L \leqslant C\|\psi\|_{\mathcal{H}} \qquad \forall \psi \in \mathcal{H}.$$

In particular, the natural embedding of $\mathcal{H}$ into $\mathcal{S}'$ is continuous.

Proof. Any $\psi \in \mathcal{H}$ can be written as

$$\begin{aligned}\psi &= \sum_{k>0} a_k\phi_k + \sum_{k>0} b_k\phi_k^* + \sum_{k<0} c_k\phi_k + \sum_{k<0} d_k\phi_k^* + \left(\alpha + \beta(\arctan u^+ + \arctan u^-)\right)\\ &= \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5.\end{aligned}$$

We will show $\psi_1 \in L$. The point is the following: it is easy to check that ϕ_k form an orthogonal set for *both* Hilbertian structures. As a result one has

$$\begin{aligned}\|\psi_1\|_L &\leqslant \frac{1}{2}\left(\sum_{k>0}\frac{1}{|k|}|a_k|^2\right)^{1/2}\\ &\leqslant \left(\sum_{k>0}|a_k|^2\right)^{1/2}\\ &= \|\psi_1\|_{\mathcal{H}}\\ &\leqslant \|\psi\|_{\mathcal{H}}\end{aligned}$$

where the last inequality is due to the orthogonality of ψ_i with respect to the $\mathcal{H}$-inner product. Similarly one obtains

$$\|\psi_i\|_L \leqslant \|\psi\|_{\mathcal{H}} \qquad \text{for} \quad i = 1, 2, 3, 4.$$

Moreover, explicit computation gives

$$\|\psi_5\|^2_{\mathcal{H}} = 2|\alpha|^2 + 8\pi^2|\beta|^2 \qquad \text{and} \qquad \|\psi_5\|^2_L = |\alpha|^2 + \frac{\pi}{6}|\beta|^2$$

which implies

$$\|\psi_5\|_L \leqslant \|\psi\|_{\mathcal{H}}.$$

The result immediately follows. □

Note that, as a consequence, the space $\mathcal{H}$ is continuously imbedded in L, so that the elements of $\mathcal{H}$ are locally integrable functions. We are now ready to prove the following result:

Proposition . *The map sending $f \in \mathcal{S}(\mathbb{R}^2)$, $\psi \in \mathcal{H}$ to $(f, \psi) \in \mathbb{C}$ is jointly continuous on $\mathcal{S}(\mathbb{R}^2) \times \mathcal{H}$ and the map $p\colon f \in \mathcal{S}(\mathbb{R}^2) \mapsto p(f) \in \mathcal{H}$ is a vector-valued distribution.*

Proof. For $f \in \mathcal{S}$ we define

$$\|f\|_\pm = \max(\|F_+\|_L, \|F_-\|_L) \qquad \text{where} \quad F_\varepsilon(u^\varepsilon) = (1 + (u^\varepsilon)^2) \int f(u^+, u^-)\, \mathrm{d}u^{-\varepsilon}.$$

We begin with ψ_1. One has

$$\begin{aligned}(f, \psi_1) &= \iint f(u^+, u^-)\psi_1(u^+)\, \mathrm{d}u^+ \,\mathrm{d}u^- \\ &= \int \psi_1(u^+)(1 + (u^+)^2) \int f(u^+, u^-)\, \mathrm{d}u^- \frac{\mathrm{d}u^+}{(1 + (u^+)^2)} \\ &= \int \psi_1(u^+) F_+(u^+) \frac{\mathrm{d}u^+}{(1 + (u^+)^2)}.\end{aligned}$$

One can easily verify that $F_+ \in L$ and

$$\begin{aligned}|(f, \psi_1)| &\leqslant \|\psi_1\|_L \|F_+\|_L \\ &\leqslant C\|\psi_1\|_{\mathcal{H}} \|f\|_\pm.\end{aligned}$$

We proceed similarly for $\psi_2, \ldots, \psi_5$ and the result follows easily. □

The field is therefore defined by

$$\varphi(f) = \frac{1}{\sqrt{2}}(a(p(f)) + a^\dagger(p(f))) \tag{7}$$

Equations (7) and (6) furnish an intrinsic, basis-free expression for the field, from which we can prove its covariance. It is indeed clear from this definition that p intertwines the natural action V of the Poincaré group both on the set of test functions and on $\mathcal{H}$; in fact we have

$$\begin{aligned}\langle V_g p(f), \psi\rangle &= \left\langle p(f), V_g^{-1}\psi\right\rangle \\ &= \left(f, V_g^{-1}\psi\right) \\ &= (V_g f, \psi) \\ &= \langle p(V_g f), \psi\rangle.\end{aligned}$$

This and the usual properties of annihilators and creators easily imply that

$$\underline{V}_g \varphi(f) \underline{V}_g^{-1} = \varphi(V_g f) \tag{8}$$

for any g in the Poincaré group, where $\underline{V}$ is the action of the Poincaré group on the Fock space $\underline{\mathcal{H}}$ (second quantization of V). That is to say that the field is Poincaré covariant.

We now turn to the conformal covariance of the field. Let us recall that the conformal group is $G_c = SO_0(2, 2)/\{\pm \mathrm{Id}\}$. One would like to obtain a property which reads formally

$$\varphi(g^{-1} \cdot x) = \underline{V}_g \varphi(x) \underline{V}_g^{-1} \qquad \text{for any} \quad g \in G_c.$$

But several difficulties appear when dealing with such a formula. First, $\mathcal{K}$ is closed under the action of the conformal group but this is not the case for $\mathcal{H}$ and hence the formula can make sense only when taking expectation values between the physical states. Second, the space of test functions is not invariant under the conformal group. Third, when smearing the distributions, one uses the Lebesgue measure which is not conformally invariant and one has to be careful when dealing with the action of the conformal group on distributions. We proceed as in [DBR2].

Let $f \in \mathcal{C}_0^\infty(M)$ be a test function and $g \in G_c$; we say that f and g are compatible if and only if there exists $X \in so(2, 2)$ such that $g = \exp X$ and $\exp\theta X \cdot x$ belongs to M for all $\theta \in [0, 1]$ and for all x in the support of f. For f and g to be compatible we define a test function $\tilde{V}_g f \in \mathcal{C}_0^\infty(M)$ by

$$\int \psi(x) \left(\tilde{V}_g f\right)(x)\, \mathrm{d}^2x = \int \psi(g \cdot x) f(x)\, \mathrm{d}^2x$$

for any locally integrable function ψ. Thanks to the compatibility condition, the right-hand side makes sense. The (local) representation $\tilde{V}$ is not the regular one (i.e., not V_g) because the measure is not conformally invariant. For example, note that for the dilation subgroup $x \mapsto \lambda x$ the compatibility condition is fulfilled and one has

$$\tilde{V}_\lambda f(x) = \frac{1}{\lambda^2} f\left(\frac{x}{\lambda}\right).$$

Let $\underline{\mathcal{K}}$ be the subspace of $\underline{\mathcal{H}}$ generated (as a tensor algebra) by $\mathcal{K}$. The representation V of the conformal group on $\mathcal{K}$ extends to a representation $\underline{V}$ of the same group on a dense domain in $\underline{\mathcal{K}}$. Then for any $f \in \mathcal{C}_0^\infty(M)$ and $g \in G_c$ to be compatible and for any ψ_1, ψ_2 in this domain one has

$$\langle \psi_1 | \varphi(\tilde{V}_g f) | \psi_2 \rangle = \langle \underline{V}_g^{-1} \psi_1 | \varphi(f) | \underline{V}_g^{-1} \psi_2 \rangle. \tag{9}$$

More precisely, one can say that $\varphi(g^{-1} \cdot x) = \underline{V}_g \varphi(x) \underline{V}_g^{-1}$ on $\underline{\mathcal{K}}$.

Sketch of the proof of (9): As for covariance, one can prove easily that $Xp = pX$ for any $X \in so(2, 2)$ (note that although $\mathcal{H}$ is not invariant under G_c, it is invariant under $so(2, 2)$ in the sense discussed at the end of section 2). The standard computation on creators and annihilators shows that $[X, \varphi(f)] = \varphi(Xf)$. Then putting $\hat{X} = -\mathrm{i}X$ one obtains

$$[\hat{X}, \varphi(f)] = -\mathrm{i}\varphi(Xf)$$

which integrates into the desired formula. We conclude that the field is covariant and conformally covariant in the sense of (8) and (9).

For later use, we compute W the distributional kernel of p, defined formally by

$$p(f)(x') = \int W(x', x) f(x)\, \mathrm{d}^2x.$$

We have

$$\begin{aligned} \langle p(f_1), p(f_2) \rangle &= (f_1, p(f_2)) \\ &= \iint f_1(x') W(x', x) f_2(x)\, \mathrm{d}^2x\, \mathrm{d}^2x' \end{aligned}$$

that is to say in the unsmeared form

$$W(x, x') = \langle p(x), p(x') \rangle.$$

Explicit computation using the basis gives

$$W(x, x') = -\mathrm{i}\tilde{G}(x, x') = -\mathrm{i}\tilde{G}(x - x')$$

where $\tilde{G} = G^{\mathrm{adv}} - G^{\mathrm{ret}}$ is the usual classical propagator [I]. Note that

$$p(x)(x') = -\mathrm{i}\tilde{G}(x - x'). \tag{10}$$

We end this section with a brief comment on the implementation of the gauge transformations. Defining

$$\gamma^\lambda = \exp\left(\mathrm{i}\lambda \frac{a(\mathrm{i}\psi_g) + a^\dagger(\mathrm{i}\psi_g)}{\sqrt{2}}\right)$$

a straightforward computation gives

$$\gamma^{-\lambda} \varphi(x) \gamma^\lambda = \varphi(x) + \lambda$$

showing that the gauge transformations are unitarily implemented in $\underline{\mathcal{H}}$.

An important feature of our construction is the appearance of negative frequency solutions in the total space and hence of corresponding terms in the field; the main problem is then the following. Does the appearance of these negative frequency solutions yield unphysical features like the appearance of negative values for the energy in physical states? In the next section, we compute the stress tensor and show that no such negative energies appear. In addition, it will turn out that the additional terms in the expression for the field give rise to a covariant renormalization: the vacuum energy vanishes without any reordering.

4. Observables and energy–momentum tensor

We are now in a position to check some of the physical features of the field we have constructed. Let $\underline{\mathcal{K}}$ be the closed subspace of $\underline{\mathcal{H}}$ generated by the $a^\dagger(\phi_1)\cdots a^\dagger(\phi_n)|0\rangle$, for $\phi_1,\ldots,\phi_n \in \mathcal{K}$. This is the space of second-quantized physical states. On this space, the inner product is positive but degenerate. Two physical states $|\Psi\rangle$ and $|\Psi'\rangle$ are said to be physically equivalent when

$$\langle\Phi|\Psi\rangle = \langle\Phi|\Psi'\rangle$$

for any physical state $|\Phi\rangle$. Let us remark that gauge changes map physical states into equivalent ones: $\gamma^\lambda\Phi$ is physically equivalent to Φ.

We call $\underline{\underline{\mathcal{N}}}$ the subspace of $\underline{\mathcal{K}}$ orthogonal to $\underline{\mathcal{K}}$

$$\Psi \in \underline{\underline{\mathcal{N}}} \text{ iff } \Psi \in \underline{\mathcal{K}} \qquad \text{and} \qquad \langle\Psi,\Phi\rangle = 0 \quad \forall\Phi \in \underline{\mathcal{K}}.$$

One has clearly that $|\Psi\rangle$ and $|\Psi'\rangle$ are physically equivalent when $|\Psi\rangle - |\Psi'\rangle \in \underline{\underline{\mathcal{N}}}$. As a consequence, the physical states, *stricto sensu*, are elements of

$$\underline{\mathcal{K}}/\underline{\underline{\mathcal{N}}} = \underline{(\mathcal{K}/\mathcal{N})}$$

where the latter is the Fock space build on the Hilbert space $\mathcal{K}/\mathcal{N}$. We have now the second-quantized Gupta–Bleuler triplet:

$$\underline{\underline{\mathcal{N}}} \subset \underline{\mathcal{K}} \subset \underline{\mathcal{H}}.$$

For physical observables, physically equivalent states must give the same mean values. As a consequence, an observable is defined to be a symmetric operator A fulfilling the following condition. When Ψ and Ψ' are equivalent physical states, then

$$\langle\Psi|A|\Psi\rangle = \langle\Psi'|A|\Psi'\rangle.$$

One can see at once that $a(\phi)+a^\dagger(\phi)$ is an observable if and only if ϕ belongs to $\mathcal{N}^\perp = \mathcal{K}+\mathcal{K}^*$. As a consequence the field itself is not an observable in general. In fact, one can prove that $\varphi(f)$ is an observable iff $\mathcal{F}f(0) = 0$, where $\mathcal{F}f$ is the Fourier transform of f. This is a manifestation of infrared divergence.

Nevertheless the operators $\partial_\mu\varphi(f)$ are observables since $\partial_\mu p(f) \in \mathcal{N}^\perp$. This is a consequence of the following computation:

$$\begin{aligned}\langle\partial_\mu p(f),\psi_g\rangle &= -\langle p(\partial_\mu f),\psi_g\rangle \\ &= -(\partial_\mu f,\psi_g) \\ &= (f,\partial_\mu\psi_g) \\ &= 0. \end{aligned} \tag{11}$$

Remark . The above definition of observables is a bit unsatisfactory, because there is no guarantee that the product of two observables is again an observable. A stronger definition is the following: a symmetric operator A is an observable iff it belongs to the operator algebra generated by the $a(\phi)$ (and $a^\dagger(\phi)$) for $\phi \in \mathcal{K}+\mathcal{K}^*$. The operators $\partial_\mu\varphi(f)$ are observables also in this sense.

At this stage, we must point out that in the rest of this section the text is not meant to be totally rigorous since we shall not worry about multiplying distributions. Among the observables, the momentum–energy tensor plays a central role. In our construction, we use negative frequency solutions as a constitutive part of the field, and one can rightfully worry whether this could lead to physical states having negative energy. *As we prove now, it turns out that physical states have positive energy*. The stress tensor is given by [BD]

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi.$$

Let us consider the excited physical state

$$|\Psi\rangle = \frac{1}{\sqrt{n_1!\ldots n_j!}}\left(a^\dagger(\psi_1)\right)^{n_1}\ldots\left(a^\dagger(\psi_j)\right)^{n_j}|0\rangle$$

where $\psi_1, \ldots, \psi_j$ are elements of $\mathcal{K}$.

In order to compute $\langle\Psi|T_{\mu\nu}(x)|\Psi\rangle$, we begin with $\langle\Psi|\partial_\mu\varphi(x)\partial_\nu\varphi(x)|\Psi\rangle$. The first remark is that $p(x)$ (or in the smeared form $p(f)$ for f real) is real as a function on the spacetime. As a consequence

$$\langle p(x), p(x)\rangle = 0.$$

This implies that $[a(p(x)), a^\dagger(p(x))] = 0$. A similar computation as (11) allows to prove that

$$\langle\partial_\mu p(x), \partial_\nu p(y)\rangle = \langle\partial_\nu p(x), \partial_\mu p(y)\rangle.$$

And of course this implies that

$$\langle\partial_\mu p(x), \partial_\nu p(x)\rangle = 0.$$

A striking consequence is that

$$[a(p(x)), a^\dagger(p(x))] = 0 \qquad \text{and} \qquad [\partial_\mu a(p(x)), \partial_\nu a^\dagger(p(x))] = 0. \quad (12)$$

Nevertheless, this does not mean that the field is in some way a classical field because $\varphi(x)$ commutes neither with $\varphi(y)$ in general nor with the usual a_k, $a_k^\dagger$, etc. A consequence of (12) is that the usual divergence in the computation of T_{00} now disappears. Then using only $[a(\phi), a^\dagger(\phi')] = \langle\phi, \phi'\rangle$, one obtains that

$$\langle\Psi|\partial_\mu\varphi(x)\partial_\nu\varphi(x)|\Psi\rangle = \sum_{i=1}^{j} n_i \mathrm{Re}(\partial_\mu\psi_i(x)\partial_\nu\psi_i^*(x)).$$

As a consequence we obtain

$$\langle\Psi|T_{00}(x)|\Psi\rangle > 0$$

and no negative energy appears in spite of the negative frequency part of the field.

5. Particle detectors

As a further check on the physical properties of the field, we investigate in this section its behaviour when coupled to a particle detector. We will show in particular that, although the field contains many unusual terms, an inertial detector cannot be excited by the vacuum: in this sense the vacuum is indeed empty. We will furthermore see that for a uniformly accelerated detector the vacuum behaves like a thermal state, as is expected.

We will follow closely the discussion of detectors in Birrell and Davies [BD]. We consider a detector moving along a world line described by the functions $x^\mu(\tau)$. Since the field φ is not an observable we rather consider the following interaction potential:

$$V(\tau) = \mathrm{e}^{\mathrm{i}H_0\tau}V_0\mathrm{e}^{-\mathrm{i}H_0\tau} \otimes \dot{x}^\mu(\tau)\partial_\mu\varphi(x(\tau))$$

where H_0 is the free Hamiltonian of the detector in its rest frame. We consider the transition between an initial state $|E_0, 0\rangle$ and a final state $\langle E, \Psi|$ where $|E\rangle$ and $|E_0\rangle$ are eigenvectors of H_0, $|0\rangle$ is the field vacuum and $|\Psi\rangle$ a physical state of the field. The amplitude for this transition may be given by the first-order perturbation theory as

$$I = \mathrm{i}\langle E, \Psi| \int_{-\infty}^{\infty} V(\tau)\,\mathrm{d}\tau |E_0, 0\rangle.$$

Clearly, only the one-particle sector component of Ψ will play a role and we can set $|\Psi\rangle = |\psi\rangle$ with $\psi \in \mathcal{K}$.

Before specifying the trajectory $x^\mu(\tau)$, and in order to make explicit computations, we write down a plane-wave decomposition of the field. We begin with some notations:

$$\xi_+ = \frac{\xi_0 - \xi_1}{2} \qquad \xi_- = \frac{\xi_0 + \xi_1}{2} \qquad x \cdot \xi = u^+\xi_+ + u^-\xi_-.$$

We are going to write the field using the functions

$$\psi_\xi(x) = \mathrm{e}^{\mathrm{i}x\cdot\xi} \qquad \psi_\xi^+(x) = \mathrm{e}^{\mathrm{i}u^+\xi_+} \qquad \text{and} \qquad \psi_\xi^-(x) = \mathrm{e}^{\mathrm{i}u^-\xi_-}.$$

Note that ψ_ξ^ε is a positive frequency solution if and only if $\xi^\varepsilon \leqslant 0$. Using

$$\tilde{G}(x) = \frac{1}{4\mathrm{i}\pi} \int \mathrm{e}^{\mathrm{i}x\cdot\xi} \left(\mathrm{v.p.}\frac{1}{\xi_+}\delta(\xi_-) + \mathrm{v.p.}\frac{1}{\xi_-}\delta(\xi_-) \right) \mathrm{d}\xi_+\,\mathrm{d}\xi_-$$

one obtains, using (10)

$$p(x) = -\frac{1}{4\pi} \int \mathrm{e}^{-\mathrm{i}u^+\xi_+} \mathrm{v.p.}\frac{1}{\xi_+} \psi_\xi^+\,\mathrm{d}\xi_+ - \frac{1}{4\pi} \int \mathrm{e}^{-\mathrm{i}u^-\xi_-} \mathrm{v.p.}\frac{1}{\xi_-} \psi_\xi^-\,\mathrm{d}\xi_-$$

and a new expression of the field:

$$\begin{aligned}\varphi(x) = &-\frac{1}{4\pi\sqrt{2}} \int \mathrm{e}^{\mathrm{i}u^+\xi_+} \mathrm{v.p.}\frac{1}{\xi_+} a(\psi_\xi^+)\,\mathrm{d}\xi_+ - \frac{1}{4\pi\sqrt{2}} \int \mathrm{e}^{-\mathrm{i}u^+\xi_+} \mathrm{v.p.}\frac{1}{\xi_+} a^\dagger(\psi_\xi^+)\,\mathrm{d}\xi_+ \\ &- \frac{1}{4\pi\sqrt{2}} \int \mathrm{e}^{\mathrm{i}u^-\xi_-} \mathrm{v.p.}\frac{1}{\xi_-} a(\psi_\xi^-)\,\mathrm{d}\xi_- - \frac{1}{4\pi\sqrt{2}} \int \mathrm{e}^{-\mathrm{i}u^-\xi_-} \mathrm{v.p.}\frac{1}{\xi_-} a^\dagger(\psi_\xi^-)\,\mathrm{d}\xi_-\end{aligned}$$

and

$$\begin{aligned}\partial_\mu\varphi(x) = &-\frac{\mathrm{i}}{4\pi\sqrt{2}} \int \partial_\mu u^+ \mathrm{e}^{\mathrm{i}u^+\xi_+} a(\psi_\xi^+)\,\mathrm{d}\xi_+ + \frac{\mathrm{i}}{4\pi\sqrt{2}} \int \partial_\mu u^+ \mathrm{e}^{-\mathrm{i}u^+\xi_+} a^\dagger(\psi_\xi^+)\mathrm{d}\xi_+ \\ &- \frac{\mathrm{i}}{4\pi\sqrt{2}} \int \partial_\mu u^- \mathrm{e}^{\mathrm{i}u^-\xi_-} a(\psi_\xi^-)\,\mathrm{d}\xi_- + \frac{\mathrm{i}}{4\pi\sqrt{2}} \int \partial_\mu u^- \mathrm{e}^{-\mathrm{i}u^-\xi_-} a^\dagger(\psi_\xi^-)\,\mathrm{d}\xi_-\end{aligned}$$

from which one easily obtains

$$\partial_\mu\varphi(x)|0\rangle = +\frac{\mathrm{i}}{4\pi\sqrt{2}} \int \partial_\mu u^+ \mathrm{e}^{-\mathrm{i}u^+\xi_+} |\psi_\xi^+\rangle\,\mathrm{d}\xi_+ + \frac{\mathrm{i}}{4\pi\sqrt{2}} \int \partial_\mu u^- \mathrm{e}^{-\mathrm{i}u^-\xi_-} |\psi_\xi^-\rangle\,\mathrm{d}\xi_-.$$

Let us now consider an inertial detector. The Poincaré covariance of the field implies that we can suppose that it is at rest at the origin of space: $u^+ = \tau = u^-$. Then the interaction amplitude I is given by

$$\begin{aligned} I &= \mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}\tau\ \langle E|V_0|E_0\rangle \langle \psi|\partial_0\varphi(x(\tau))|0\rangle \mathrm{e}^{\mathrm{i}\tau(E-E_0)} \\ &= -\frac{1}{4\pi\sqrt{2}} \langle E|V_0|E_0\rangle \sum_{\varepsilon\in\{+,-\}} \int_{-\infty}^{+\infty} \mathrm{d}\tau \int \mathrm{d}\xi_\varepsilon\ \mathrm{e}^{-\mathrm{i}\tau\xi_\varepsilon} \left\langle \psi|\psi_\xi^\varepsilon \right\rangle \mathrm{e}^{\mathrm{i}\tau(E-E_0)} \\ &= -\frac{1}{2\sqrt{2}} \langle E|V_0|E_0\rangle \sum_{\varepsilon\in\{+,-\}} \int \mathrm{d}\xi_\varepsilon\ \delta(E - E_0 - \xi_\varepsilon) \left\langle \psi|\psi_\xi^\varepsilon \right\rangle. \end{aligned}$$

If $E > E_0$, we expect $I = 0$, since the vacuum should not be able to excite the detector when the latter follows an inertial trajectory. This is indeed the case since $\langle\psi|\psi_\xi^\varepsilon\rangle = 0$ whenever $\psi \in \mathcal{K}$ and $\xi_\varepsilon > 0$. Indeed, $\mathcal{K}$ consists of positive frequency solutions of the equation and ψ_ξ^ε is negative frequency for $\xi_\varepsilon > 0$.

We continue with a uniformly accelerated detector:

$$\begin{cases} x^0(\tau) = \alpha \sinh(\tau/\alpha) \\ x^1(\tau) = \alpha \cosh(\tau/\alpha). \end{cases}$$

and choose the physical state

$$\psi(x) = \psi_k(x) = \mathrm{e}^{-\mathrm{i}(|k|x^0 - kx^1)}.$$

This yields the following expression for the transition amplitude I_k:

$$\begin{aligned} I_k &= \mathrm{i}\int \mathrm{d}\tau\, \langle E|V_0|E_0\rangle \dot{x}^\mu(\tau)\langle\psi_k|\partial_\mu\varphi(x(\tau))|0\rangle \mathrm{e}^{\mathrm{i}\tau(E-E_0)} \\ &= \mathrm{i}\langle E|V_0|E_0\rangle \int d\tau\, \mathrm{e}^{\mathrm{i}\tau(E-E_0)} \frac{\mathrm{d}}{\mathrm{d}\tau}\langle\psi_k|\varphi(x(\tau))|0\rangle \\ &= \langle E|V_0|E_0\rangle(E-E_0)\int \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\tau(E-E_0)}\langle\psi_k|\varphi(x(\tau))|0\rangle. \end{aligned}$$

Neglecting the (strong) divergence of such an integral, we continue by computing the total probability for a transition to occur, that is to say the sum of all transition probabilities $|I_k|^2$:

$$\begin{aligned} \int \frac{\mathrm{d}k}{|k|}|I_k|^2 &= |\langle E|V_0|E_0\rangle|^2(E-E_0)^2 \\ &\times \left|\iint \mathrm{d}\tau\,\mathrm{d}\tau'\, \mathrm{e}^{\mathrm{i}(\tau-\tau')(E-E_0)} \int \frac{\mathrm{d}k}{|k|}\langle 0|\varphi(x(\tau'))|\psi_k\rangle\langle\psi_k|\varphi(x(\tau))|0\rangle\right|^2. \end{aligned}$$

Note that the ψ_k's are a complete system of *physical* states, and not of all states; as a consequence

$$\int \frac{\mathrm{d}k}{|k|}\langle 0|\varphi(x)|\psi_k\rangle\langle\psi_k|\varphi(y)|0\rangle \neq \langle 0|\varphi(x)\varphi(y)|0\rangle.$$

In fact, explicit computation, using regularization of Fourier integrals, gives

$$\int \frac{\mathrm{d}k}{|k|}\langle 0|\varphi(x)|\psi_k\rangle\langle\psi_k|\varphi(y)|0\rangle = -\frac{1}{2}(\mathrm{Log}\left((x-y)^2 + \mathrm{i}\varepsilon(x-y)^0\right) =: 2\pi W_\kappa(x-y)$$

where Log is the principal determination of the logarithm. The above term is a function of $\Delta(\tau) = (\tau' - \tau)$. More precisely, we obtain the following expression for the transition probability per unit time:

$$\begin{aligned} &\frac{1}{4}|\langle E|V_0|E_0\rangle|^2(E-E_0)^2\left|\int \mathrm{d}(\Delta\tau)\,\mathrm{e}^{\mathrm{i}(\Delta\tau)(E-E_0)}\left(\ln\left(2\cosh\frac{\Delta\tau}{\alpha} - 2\right) + \mathrm{i}\pi\,\mathrm{sgn}(\Delta\tau)\right)\right|^2 \\ &\qquad = \frac{1}{4}|\langle E|V_0|E_0\rangle|^2|(E-E_0)\hat{f}(E-E_0) - 2\pi|^2 \end{aligned}$$

where $\hat{f}$ is the Fourier transform of a function f verifying

$$f(\Delta\tau + \mathrm{i}\alpha) = f(\Delta\tau).$$

Following [BD], one can then recognize the appearance of a thermal bath at the temperature

$$T = 1/2\pi\alpha k_\mathrm{B}$$

where k_B is the Boltzmann constant.

6. The Morchio *et al* field revisited

As explained in detail in [DBR2], the obstruction to building a massless free quantum field theory on two-dimensional spacetimes can be seen as follows: the Klein–Gordon inner product is degenerate on the natural one-particle physical space $\mathcal{K}$. The point of view in this paper as well as in [DBR2] is that one can overcome this obstruction rather straightforwardly by using a Gupta–Bleuler construction with a suitably chosen invariant and non-degenerate total space containing $\mathcal{K}$ (and also some non-physical states) and then by proceeding to canonical quantization. The space considered in both cases contains some negative frequency states, but we proved this does not lead to unusual physical features for the field.

For the (1 + 1)-dimensional Minkowski spacetime, a massless Poincaré-covariant scalar quantum field has been constructed using an *a priori* quite different approach in [MPS]. We recall this construction briefly below and will then show that the resulting field can equally well be obtained through a straightforward application of the above Gupta–Bleuler approach using the triplet

$$\mathcal{N} \subset \mathcal{K} \subset \mathcal{H}_K$$

where only the total space $\mathcal{H}_K$ differs from the one used in this paper. On $\mathcal{H}_K$ the inner product is non-degenerate but non-definite (there is one 'negative' direction) and $\mathcal{H}_K$ is Poincaré-invariant but not conformally invariant. The usual canonical quantization on this space leads in a simple manner to a Poincaré covariant quantum field which is precisely the field of [MPS]. This field is however not conformally invariant.

The construction in [MPS] starts with the Poincaré (but not conformally) covariant two-point function

$$W_\kappa(x) = -\frac{1}{4\pi}\mathrm{Log}(x^2 + \mathrm{i}\varepsilon x^0) = -\frac{1}{4\pi}\ln|x^2| - \frac{\mathrm{i}}{2}\tilde{G}(x)$$

that we encountered in our discussion of the accelerated detector. Using W_κ, one defines a sesquilinear form

$$\langle f, g\rangle_\kappa = \int f^*(x)g(y)W_\kappa(x-y)\,\mathrm{d}^2x\,\mathrm{d}^2y \qquad \forall f, g \in \mathcal{S}$$

on the Schwartz space of test functions $\mathcal{S}$. This form is both degenerate and non-definite. An explicit computation shows readily that for all $f \in \mathcal{S}_0 = \{f \in \mathcal{S} \mid \mathcal{F}f(0) = 0\}$ and $g \in \mathcal{S}$ one has

$$\langle f, g\rangle_\kappa = \pi\int_{\mathbb{R}} \frac{\mathrm{d}\xi_1}{|\xi_1|}\mathcal{F}f(-|\xi_1|, \xi_1)^*\mathcal{F}g(-|\xi_1|, \xi_1). \tag{13}$$

Here

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}^2x.$$

Note that in these notations 'positive frequency' components correspond to $\xi_0 \leqslant 0$. Quotienting $\mathcal{S}$ by the Wightman ideal

$$\mathcal{I} = \{f \in \mathcal{S} \mid \langle f, g\rangle_\kappa = 0, \forall g \in \mathcal{S}\}$$

one obtains the space $S = \mathcal{S}/\mathcal{I}$ on which now $\langle\cdot,\cdot\rangle_\kappa$ is non-degenerate, but still non-definite. In order to construct the one-particle sector of the field, the authors of [MPS] equip S with a Hilbert topology dominating $\langle\cdot,\cdot\rangle_\kappa$, as follows. Let $\chi \in \mathcal{S}$ so that $\mathcal{F}\chi(0) = 1$, $\langle\chi, \chi\rangle_\kappa = 0$ and define, for all $f \in \mathcal{S}$

$$f_0 = f - \mathcal{F}f(0)\chi$$

and

$$(f,g)_K = \langle f_0, g_0\rangle_\kappa + \langle f,\chi\rangle_\kappa\langle\chi,g\rangle_\kappa + (\mathcal{F}f(0))^*\,\mathcal{F}g(0).$$

One can check that $(\cdot,\cdot)_K$ defines a positive definite sesquilinear form on S and that

$$|\,\langle f,g\rangle_\kappa\,| \leqslant ||f||_K||g||_K. \tag{14}$$

The one-particle sector of the theory is finally obtained by completing S with respect to this pre-Hilbert structure:

$$\mathcal{K}^{(1)} = \overline{S}^K.$$

Note that (14) implies that $\langle\cdot,\cdot\rangle_\kappa$ extends to a bi-continuous sesquilinear form on $\mathcal{K}^{(1)}$. The field of [MPS] is finally constructed in the usual way by considering the Fock space over $\mathcal{K}^{(1)}$.

In order to compare this construction to our approach, we need a better understanding of the space $\mathcal{K}^{(1)}$. The above construction of the space $\mathcal{K}^{(1)}$ is rather abstract and the use of the auxiliary function χ is not very satisfactory. For a free field one would expect that the one-particle space should be realizable as a space of (distributional) solutions of the classical field equation, a fact that is not obvious from the above abstract construction. We will show below that, contrary to a claim in [MPS], $\mathcal{K}^{(1)}$ can indeed be identified very naturally with such a space (called $\mathcal{H}_K$ below) and that this space does not depend on the choice of χ. We will furthermore see that it is the total space of a Gupta–Bleuler triplet

$$\mathcal{N}\subset\mathcal{K}\subset\mathcal{H}_K$$

where $\mathcal{N}$ and $\mathcal{K}$ are precisely the spaces introduced in section 2.

First it will be useful to describe the structure of the abstract space $\mathcal{K}^{(1)}$ in some more detail. We define the subspace $\mathcal{K}'^{(1)} = \overline{S_0}^K$, where $S_0 = \mathcal{S}_0/\mathcal{I}$ and $\mathcal{S}_0 = \{f\in\mathcal{S}\mid \mathcal{F}f(0)=0\}$. Note that $\mathcal{K}^{(1)}$ contains furthermore two distinguished one-dimensional subspaces, defined as follows. The first one is $V=\mathbb{C}\chi$, the second is $V_0=\mathbb{C}v_0$, where $v_0\in\mathcal{K}^{(1)}$ is defined via

$$(v_0,\psi)_K = \langle\chi,\psi\rangle_\kappa \qquad \forall\psi\in\mathcal{K}^{(1)}.$$

It can be shown (see [MPS]) that $(v_0,v_0)_K = 1$ so that $\langle v_0,\chi\rangle_\kappa = 1$ and that $v_0\in\overline{S_0}^K\backslash S_0$. Moreover, for all $f\in\mathcal{S}$

$$\langle v_0,\dot f\rangle_\kappa = \mathcal{F}f(0) \tag{15}$$

where $\dot f$ is the class of f in $S\subset\mathcal{K}^{(1)}$ (we shall not hesitate to drop the dot if no confusion arises). In terms of these subspaces, $\mathcal{K}^{(1)}$ admits the following $(\cdot,\cdot)_K$-orthogonal direct sum decomposition

$$\mathcal{K}^{(1)} = \mathcal{K}'^{(1)}\oplus V.$$

We are now ready to identify $\mathcal{K}^{(1)}$ with a space of distributional solutions of the classical field equation, as follows. First define, for all $\psi\in\mathcal{K}^{(1)}$, $f\in\mathcal{S}$ real,

$$T^{(1)}_\psi(f) = \langle\dot f,\psi\rangle_\kappa.$$

Since $|T^{(1)}_\psi(f)|\leqslant\|\dot f\|_K\|\psi\|_K$, it is easily checked that the map $\psi\mapsto T^{(1)}_\psi$ is continuous from $\mathcal{K}^{(1)}$ to $\mathcal{S}'$. The non-degeneracy of $\langle\cdot,\cdot\rangle_\kappa$ on $\mathcal{K}^{(1)}$ guarantees furthermore that $T^{(1)}$ is injective and we define

$$\mathcal{H}_K \equiv T^{(1)}\mathcal{K}^{(1)}.$$

The inner product $\langle\,,\,\rangle_\kappa$ is covariant, as a consequence $T^{(1)}$ intertwines the natural representations of the Poincaré group on $\mathcal{K}^{(1)}$ and $\mathcal{H}_K$. For $f\in S$, it is clear that

$$T^{(1)}_{\dot f} = W_\kappa\star f.$$

Hence $T_f^{(1)}$ is indeed a distributional solution of the field equation. The continuity of the map $T^{(1)}$ guarantees then, by a density argument, that $T_\psi^{(1)}$ is a distributional solution of the field equation for any $\psi \in \mathcal{K}^{(1)}$. Much more can actually be said.

Proposition. We have

$$T^{(1)}V_0 = \mathcal{N} \qquad \text{and} \qquad T^{(1)}\mathcal{K}'^{(1)} = \mathcal{K}.$$

Moreover, $T^{(1)}$ defines a bounded linear bijection between the Hilbert spaces $\mathcal{K}'^{(1)}$ and $\mathcal{K} \subset \mathcal{H}$ and intertwines the natural representations of the Poincaré group on these spaces. Also, $T^{(1)}\mathcal{K}^{(1)}$ is a space of tempered distributions that does not depend on the choice of χ used in its construction.

Proof. The first statement is immediate from (15). The proof of the second requires a little more work. We will first show through an explicit computation that, for $\dot{f} \in S_0$, $T_{\dot{f}}^{(1)} \in \mathcal{K}$. Using the definition it is easy to see that for $f \in S_0$

$$T_f^{(1)} = \frac{\pi}{2}\mathcal{F}^{-1}\left(\left(\frac{1}{\xi_+}\right)_- \delta(\xi_-)\mathcal{F}f(\xi) + \left(\frac{1}{\xi_-}\right)_- \delta(\xi_+)\mathcal{F}f(\xi)\right).$$

A further computation then yields

$$T_{\dot{f}}^{(1)}(x) = -\frac{1}{2}\int_{-\infty}^{0}\frac{\mathrm{d}\xi_+}{\xi_+}\,\mathrm{e}^{iu^+\xi_+}\,\mathcal{F}f(\xi_+, -\xi_+) - \frac{1}{2}\int_{-\infty}^{0}\frac{\mathrm{d}\xi_-}{\xi_-}\,\mathrm{e}^{\mathrm{i}u^-\xi_-}\,\mathcal{F}f(\xi_-, \xi_-)$$

from which one easily sees that $T_{\dot{f}}^{(1)}$ belongs to $\mathcal{K}$. We now show $T^{(1)}$ is actually continuous as a map from S_0 to $\mathcal{K}$, viewed as a Hilbert subspace of $\mathcal{H}$. For that purpose we need to compute $\|T_{\dot{f}}^{(1)}\|_{\mathcal{H}}$. With the notations of section 2, and using the definition of $\mathcal{K}$ it is clear that, $\forall\psi \in \mathcal{K}$

$$\|\psi\|_{\mathcal{H}}^2 = |\langle\psi,\psi\rangle| + \tfrac{1}{2}|c_g|^2.$$

To proceed, we need an explicit expression for c_g. We have

$$\psi(x^0, x^1) = c_g + \sum_{k>0} c_k\phi_k(u^+) + \sum_{k<0} d_k\phi_k(u^-) = c_g + \phi_+(u^+) + \phi_-(u^-).$$

Note that the sums converge absolutely on $\mathcal{P}^\epsilon$ (so that $c_g = \psi(-\mathrm{i}, 0)$) and in the sense of L. It is then easy to see that $\int_{\mathbb{R}} \phi^\epsilon(u^\epsilon)\frac{\mathrm{d}u^\epsilon}{1+(u^\epsilon)^2} = 0$ from which one concludes that

$$c_g = \frac{1}{\pi}\int_{\mathbb{R}} \psi(0, x^1)\frac{\mathrm{d}x^1}{1 + (x^1)^2}.$$

Since

$$T_{\dot{f}}^{(1)}(0, x^1) = \frac{1}{2}\int_{\mathbb{R}}\frac{\mathrm{d}\xi_1}{|\xi_1|}\,\mathrm{e}^{\mathrm{i}x^1\xi_1}\mathcal{F}f(-|\xi_1|, \xi_1)$$

one can compute c_g for $\psi = T_{\dot{f}}^{(1)}$ to obtain finally

$$c_g = \frac{\pi}{2}\int_{\mathbb{R}}\frac{\mathrm{d}\xi_1}{|\xi_1|}\,\mathrm{e}^{-|\xi_1|}\,\mathcal{F}f(-|\xi_1|, \xi_1).$$

Now we estimate c_g as follows.

$$\begin{aligned}
\frac{2}{\pi}|c_g| &\leqslant \left|\int_{\mathbb{R}}\frac{\mathrm{d}\xi_1}{|\xi_1|}(\mathrm{e}^{-|\xi_1|} - \mathcal{F}\chi(-|\xi_1|, \xi_1))\mathcal{F}f(-|\xi_1|, \xi_1)\right| \\
&\quad + \left|\int_{\mathbb{R}}\frac{\mathrm{d}\xi_1}{|\xi_1|}\,\mathcal{F}\chi(-|\xi_1|, \xi_1))\mathcal{F}f(-|\xi_1|, \xi_1)\right| \\
&\leqslant \left[\int_{\mathbb{R}}\frac{\mathrm{d}\xi_1}{|\xi_1|}|\mathrm{e}^{-|\xi_1|} - \mathcal{F}\chi(-|\xi_1|, \xi_1)|^2\right]^{1/2}\langle\dot{f}, \dot{f}\rangle_\kappa^{1/2} + \frac{1}{\pi}|\langle\dot{\chi}, \dot{f}\rangle_\kappa| \\
&\leqslant C\|\dot{f}\|_K.
\end{aligned}$$

We used for the last inequality that $(\exp(-|\xi_1|) - \mathcal{F}\chi(-|\xi_1|, \xi_1))/|\xi_1| \in L^2(d\xi_1)$ since $\mathcal{F}\chi(0) = 1$. This shows that $T^{(1)}$, defined on S_0, extends to a continuous linear map $\tilde{T}^{(1)}$ from $\mathcal{K}'^{(1)}$ to $\mathcal{K}$. Since, as shown in section 3, the natural imbedding of $\mathcal{K}$ into $\mathcal{S}'$ is continuous, it follows that $\tilde{T}^{(1)}$ coincides with $T^{(1)}$ on $\mathcal{K}'^{(1)}$. It remains to show that $T^{(1)}$ maps $\mathcal{K}'^{(1)}$ onto $\mathcal{K}$. We do this in the following rather roundabout way. Define first

$$I: S_0 \to L^2(\mathbb{R}, d\xi_1/|\xi_1|) \qquad I(f) = \mathcal{F}f(-|\xi_1|, \xi_1).$$

Thanks to (13), I is clearly continuous for the $\|\cdot\|_K$-norm and hence extends to a bounded map on $\mathcal{K}'^{(1)}$, that we shall also denote by I. We will show that Ker $I = V_0$ and that I is surjective. For that purpose, first define, for $\psi = \psi_+ + \psi_- \in \mathcal{K}$:

$$T\psi(\xi_1) = \sqrt{\frac{2}{\pi}}\left(\xi_1\hat{\psi}_+(-\xi_1) - \xi_1\hat{\psi}_-(\xi_1)\right) \in L^2(\mathbb{R}, d\xi_1/|\xi_1|)$$

where ˆ denotes the one-dimensional Fourier transform:

$$\hat{\psi}(\lambda) = \frac{1}{\sqrt{2\pi}}\int e^{-iu^+\lambda}\psi(u^+)\, du^+.$$

The operator T is continuous, surjective and has the space $\mathcal{N}$ as its kernel [DBR1]. Moreover, one easily checks that, on S_0

$$T \circ T^{(1)} = I \tag{16}$$

a relation that therefore extends by continuity to all of $\mathcal{K}'^{(1)}$. Since $T^{(1)}V_0 = \mathcal{N}$, one concludes from the injectivity of $T^{(1)}$ that Ker $I = V_0$. In order to prove that I is surjective, note first that

$$\langle\psi, \phi\rangle_\kappa = (\psi, \phi)_K - (\psi, v_0)_K(v_0, \phi)_K \qquad \forall\psi, \phi \in \overline{S_0}^K$$

so that I is clearly an isometry on $V_0^\perp$, the $\|\cdot\|_K$-orthogonal complement of V_0 in $\mathcal{K}'^{(1)}$. As a result, $I(V_0^\perp)$ is closed. On the other hand, for all $f \in \mathcal{S}_0$, $I(f) = I(f - (f, v_0)_K v_0)$ so that $I(S_0) \subset I(V_0^\perp)$. But $I(S_0)$ is dense, so $I(V_0^\perp) = L^2(\mathbb{R}, d\xi_1/|\xi_1|)$. It is now easy to show that $T^{(1)}$ maps $\mathcal{K}'^{(1)}$ onto $\mathcal{K}$. Indeed, let $\psi \in \mathcal{K}$. Then there exists a $\phi \in \mathcal{K}'^{(1)}$ so that $I\phi = T\psi$ and hence $T(T^{(1)}\phi - \psi) = 0$, so that there exists $\lambda \in \mathbb{C}$ satisfying $T^{(1)}\phi = \psi + \lambda T^{(1)}v_0$. Hence ψ belongs to $T^{(1)}\mathcal{K}'^{(1)}$. The final statement of the proposition is now obvious: if χ and χ' are two different test functions used for the construction, then we have $\mathcal{H}_K = \mathcal{K} \oplus \mathbb{C}T^{(1)}_\chi$ and $\mathcal{H}'_K = \mathcal{K} \oplus \mathbb{C}T^{(1)}_{\chi'}$. But $T^{(1)}_{\chi'} - T^{(1)}_\chi$ belongs to $\mathcal{K}$ since $\chi - \chi' \in \mathcal{S}_0$, so that $\mathcal{H}_K = \mathcal{H}'_K$. □

The above identification of the [MPS] one-particle space as the space $\mathcal{K}$ shows it carries a perfectly natural global action of the conformal group, a fact that seems to have escaped the attention of the authors of [MPS]. It remains to investigate the image $\mathcal{H}_K$ of the total space $\mathcal{K}^{(1)}$ under $T^{(1)}$ and its relation to the total space $\mathcal{H}$. One can first remark that $\mathcal{H}_K \not\subset \mathcal{H}$ since $\mathcal{H}_K$ contains elements which are positive frequency and not in $\mathcal{K}$, while there are no such elements in $\mathcal{H}$. Note that on $\mathcal{H}_K$ two inner products exist: the Klein–Gordon inner product and the inner product $\langle\ ,\ \rangle_\tau$ transported from $\mathcal{K}^{(1)}$ by $T^{(1)}$:

$$\left\langle T^{(1)}_\psi, T^{(1)}_\phi\right\rangle_\tau = \langle\psi, \phi\rangle_\kappa.$$

It is not difficult in view of (16) to prove that these inner products are equal on $\mathcal{K}$ *but this is not the case on* $\mathcal{H}_K$. We shall prove this fact by showing that the inner product $\langle\ ,\ \rangle_\tau$ is not conformally invariant and not even invariant under dilations. This explains why the field of [MPS] cannot be conformally invariant in the strong sense described above and why conformal invariance can only be saved in the [MPS] theory through the use of artificial correction terms.

In order to prove our claim we consider the dilation operator V_λ by $V_\lambda\phi(x) = \phi(\lambda x)$, and first remark that we have

$$\left(T_f^{(1)}, g\right) = \langle f, g\rangle_\kappa = \left\langle T_f^{(1)}, T_g^{(1)}\right\rangle_\tau.$$

Hence, for any $\psi \in \mathcal{H}_K$:

$$\left\langle T_g^{(1)}, \psi\right\rangle_\tau = (g, \psi).$$

Let f, g be real test functions, then

$$\begin{aligned}\left(V_\lambda T_f^{(1)}, g\right) &= \frac{1}{\lambda^2}\int f(x)W_\kappa(x-y)g\left(\frac{y}{\lambda}\right) \mathrm{d}^2x\,\mathrm{d}^2y \\ &= \lambda^2\int f(\lambda x)W_\kappa(x-y)g(y)\,\mathrm{d}^2x\,\mathrm{d}^2y - 2\pi\ln\lambda\,\mathcal{F}f(0)\,\mathcal{F}g(0) \\ &= \left(T^{(1)}_{\lambda^2 V_\lambda f - 2\pi\ln\lambda\,\mathcal{F}f(0)v_0}, g\right).\end{aligned}$$

As a result, $V_\lambda T_f^{(1)} \in \mathcal{H}_K$ and

$$V_\lambda T_f^{(1)} = T^{(1)}_{\lambda^2 V_\lambda f - 2\pi\ln\lambda\,\mathcal{F}f(0)v_0}$$

and consequently

$$\left\langle V_\lambda T_f^{(1)}, V_\lambda T_g^{(1)}\right\rangle_\tau = \left\langle T_f^{(1)}, T_g^{(1)}\right\rangle_\tau - 2\pi\,\ln\lambda\,\mathcal{F}f(0)\,\mathcal{F}g(0).$$

It is now clear that $(\mathcal{H}_K, \langle\ ,\ \rangle_\tau)$ is not invariant under dilations and therefore it is not conformally invariant.

In conclusion the two first terms of the Gupta–Bleuler triplets are identical in both theories, but the third one is not. We finish this section by showing that the field constructed with this triplet using the Gupta–Bleuler construction of section 3 is precisely the field of [MPS].

Suppose therefore that one wants to proceed to canonical quantization, in complete analogy with what was done in section 3 (see formula (5)), but using $\mathcal{H}_K$ as the one-particle sector. One chooses a basis ψ_k of $\mathcal{H}_K$ orthogonal with respect to $\langle\ ,\ \rangle_\tau$ such that

$$\langle\psi_k, \psi_k\rangle_\tau = 1 \quad \text{for } k > 0 \qquad \text{and} \qquad \langle\psi_{-1}, \psi_{-1}\rangle_\tau = -1.$$

Then one defines the field via

$$\begin{aligned}\varphi(x) &= \sum_{k>0}\psi_k(x)a_k + \sum_{k>0}\psi_k^*(x)a_k^\dagger - \psi_{-1}(x)a_{-1} - \psi_{-1}^*(x)a_{-1}^\dagger \\ &= \sum_k \operatorname{sgn}(k)\psi_k(x)a_k + \sum_k \operatorname{sgn}(k)\psi_k^*(x)a_k^\dagger\end{aligned}$$

where the a_k and the $a_k^\dagger$ are the usual annihilation and creation operators of the modes ψ_k: $([a_k, a_{k'}^\dagger] = \langle\psi_k, \psi_{k'}\rangle_\tau)$. To be more precise, one has to smear the field with a real test function $f \in \mathcal{S}$:

$$\begin{aligned}\varphi(f) &= \int f(x)\varphi(x)\,\mathrm{d}^2x \\ &= \sum_k\int \operatorname{sgn}(k)\psi_k(x)f(x)\,\mathrm{d}^2x\ a_k + \sum_k\int \operatorname{sgn}(k)\psi_k^*(x)f(x)\,\mathrm{d}^2x\ a_k^\dagger \\ &= \sum_k \operatorname{sgn}(k)(\psi_k^*, f)a_k + \sum_k \operatorname{sgn}(k)(\psi_k, f)a_k^\dagger\end{aligned}$$

where the round brackets designate the L^2-inner product (4). The operators a and $a^\dagger$ are respectively antilinear and linear in the argument ψ_k. Hence we can rewrite the smeared field in the following manner:

$$\begin{aligned}\varphi(f) &= \sum_k \operatorname{sgn}(k)(\psi_k^*, f)a(\psi_k) + \sum_k \operatorname{sgn}(k)(\psi_k, f)a^\dagger(\psi_k) \\ &= a\left(\sum_k \operatorname{sgn}(k)(\psi_k, f)\psi_k\right) + a^\dagger\left(\sum_k \operatorname{sgn}(k)(\psi_k, f)\psi_k\right).\end{aligned}$$

Defining

$$p_\kappa(f) = \sum_k \operatorname{sgn}(k)(\psi_k, f)\psi_k$$

we have

$$\varphi(f) = a\,(p_\kappa(f)) + a^\dagger(p_\kappa(f)).$$

It is then readily seen that p_κ is the *unique* vector in $\mathcal{H}_K$ for which

$$\langle p_\kappa(f), \psi\rangle_\tau = (f, \psi) \qquad \forall \psi \in \mathcal{H}_K$$

so that $T_f^{(1)} = p_\kappa(f)$. As a consequence, the field we have just described admits W_κ as the Wightman function, and is therefore nothing but the field of [MPS].

As we have claimed in the beginning of this section, the field of [MPS] is the field one obtains by Gupta–Bleuler canonical quantization on a Poincaré invariant, but not conformally invariant, one-particle sector containing $\mathcal{K}$ as the physical subspace and one 'negative' direction.

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